Computer Science II-Numerical Methods

Mid Sem, March 2, 2018

Each question carries 5 marks.

1. The trace of a square matrix A is denoted by $tr(A)$ and is defined as the sum of diagonal elements of A. Write a one line Octave expression to compute the trace of a matrix A. Do not use any for – end loops in your expression.

2. Find the bug in the function demobug(n) below and fix it without dismantling the break command.

```
function k=demobug(n)
x = rand(1, n); % return a matrix with random elements uniformly distributed on the interval (0, 1).
k = 1;
while k <= n
if x(k) > 0.6
break
end
k = k + 1;
end
fprintf('x(k)= %f for k=%d n=%d ,'x(k),k,n);
```

3. Write a newsqrt function in Octave to compute the square root of a positive number using Newton's method.

4. Let $g : [a, b] \to [a, b]$ and $g \in C^1[a, b]$ with $|g'(x)| < 1$ for $x \in [a, b]$. Let $p_0 \in [a, b]$ and $p_k = g(p_{k-1}), k \geq 1$. Show that there exists $p \in [a, b]$ such that $\lim_{k\to\infty} p_k = p$ and $g(p) = p$.

5. Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in interval $[1, 4]$.

6. Consider the function $f = sign(x)\sqrt{|x|}$. Show that the Newton's method does not converge to the root for any initial guess p_0. Explain why it does not converge.

7. Let $f(x) = e^x - x - 1$. The Newton's method with $p_0 = \frac{1}{2}$ converges to a zero of $f(x)$ but not quadratically. Modify Newton's method to improve the rate of this convergence.

8. Define condition number $k(A)$ of a matrix A. Find $k(A)$ when A is given by

$$\begin{bmatrix} 2 & 1 \\ 2+\delta & 1 \end{bmatrix}$$

where δ is a small parameter and $||A||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$.

9. Suppose the data $(x_i, f(x_i)), 0 \leq i \leq n$, is given. Recall that Lagrange's interpolation polynomial is given by $\varphi_n(x) = \sum_{i=0}^{n} L_i(x)f(x_i)$ where $L_i(x) = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}$. Show without direct computation that $\sum_{i=0}^{n} L_i(x) = 1$.

10. If $f(x) = x^n$, then for $m \leq n$, prove that the m-th divided difference $f[x_0, x_1, \ldots, x_m] = \sum x_0^{\lambda_0} x_1^{\lambda_1} \cdots x_m^{\lambda_m}$ where $\lambda_0 + \lambda_1 + \ldots + \lambda_m = n - m$ and $\lambda_i \geq 0$ are integers. Deduce that the n-th divided difference of a polynomial of degree n is its leading coefficient.

11. (a). If $\Delta f(x) = f(x+h) - f(x), h > 0$, show that $\Delta^{n+1} p(x) = 0$ where $p(x)$ is a polynomial of degree $\leq n$.

(b). Consider the following table

x	0	1	2	4
$f(x)$	1	3	9	81

Let $\varphi(x)$ denote the interpolating polynomial of degree ≤ 3 that interpolates f at the given set of points. Find $\varphi(3)$ without explicitly computing $\varphi(x)$. (Hint: Use part (a)).